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On color difference formulae

Color perception belongs to an area called Psychophysics, which is defined as the scientific study of relationships between the physical measurements of stimuli and the sensations, and perceptions that those stimuli evoke. This definition implies that color perception is a physiological response to light stimuli, but not restricted solely to light. In fact other stimuli experienced at the same time induce different sensations and modulate the overall perception of color.
This psychophysical basis of color perception (also called psychochromatic perception) introduces a factor of unpredictability when conducting experiments under uncontrolled conditions and demands special care when expressing color and color differences as numbers.

Describing color

One should distinguish between basic and advanced colorimetry:

- *basic colorimetry* makes predictions whether two visual stimuli with different spectral distributions will match under given conditions of observations
- *advanced colorimetry* assesses the appearance of color stimuli presented to the observer in a normal environment.

Basic colorimetry deals with the phenomena of color of single objects as normally encountered in industrial production lines, its application however is limited to the comparison of two objects, normally one regarded as standard.

The description of color is based on the interaction of three elements:

- Light source
- Object
- Observer

Any description of color will involve these three elements that are normally regarded as unique and physically isolated. Advanced colorimetry no longer isolates these elements, but integrates them into a color appearance model, the ultimate goal of color description.
To overcome the difficulty of the task of describing colors, procedures and methods have been developed using daily experience and intuition, these have been recorded in a series of International Standards, which assure a common language when dealing with color description and color differences. It is important that the following elements are observed while assessing color and color differences:

- Chromatic adaptation
- Light adaptation
- Luminance level
- Background color
- Surround color

Mathematical description of color

Out of the elements leading to color perception the response of the observer is certainly the most important part and is based on the human visual system. Because a description of this response from first principles is extremely complex, one has opted for modeling the visual responses of a large number of persons. This results in three color-matching functions; these functions can be used to predict the visual response of human beings. In general two different observers are used according to the standards of the Commission Internationale de l'Eclairage (CIE): the 2° observer of 1931 and the 10° observer of 1964.

Objects are described by radiance factors of the material, normally approximated by its reflectance spectrum. Notice that reflectance factors, as measured by a spectrophotometer, are ratio measurements and are thus independent of the light source of the instrument at the moment of the measurement. This is however no longer the case when the material shows a certain level of fluorescence; in fact fluorescence levels depend on the amount and spectral distribution of the incident light, this leads to a further restriction of the use of spectral values for color computations.

Light sources have been standardized to a large extent by international organizations. The CIE recommends the use of daylight D_{65} and incandescent sources A. In practice many other light sources are encountered, especially when dealing with artificial lighting of working environments. Spectral distribution of such sources, if not normalized is specified normally by manufacturers.

Any mathematical description of color starts with the computation of the tri-stimulus values defined as:

$$X = \sum_{370}^{780} S_\lambda \cdot R_\lambda \cdot x_\lambda$$

$$Y = \sum_{370}^{780} S_\lambda \cdot R_\lambda \cdot y_\lambda$$

$$Z = \sum_{370}^{780} S_\lambda \cdot R_\lambda \cdot z_\lambda$$

where S_λ is the spectral distribution of the light source used during observation, R_λ is the reflectance spectrum of the sample and (x_λ, y_λ, z_λ) are the color matching functions describing the observer.

In order to represent color as a three dimensional volume, tri-stimulus values are transformed into chromaticity coordinates according to:

$$x = \frac{X}{X+Y+Z}$$

$$y = \frac{Y}{X+Y+Z}$$

$$z = \frac{Z}{X+Y+Z}$$

it is clear that the following condition holds:

$$x + y + z = 1$$

implying that only two of the quantities must be calculated, since the third depends on the values of the other two.

The system describes a color by assigning three numbers (X,Y,Z) and achieves the goal of the basic colorimetry giving information about the equivalence of two colors: in general two colors having the same set of (X,Y,Z) values are said to be equivalent and will be perceived as equal under the observation conditions used during the determination. A difficulty arises from the fact that there is no direct relationship between the values of (X,Y,Z) and the usual language used by colorists. There have been many attempts to transform the values of (X,Y,Z) into different expressions that are connected to the way of color assessment, for example the Helmholtz coordinates:

- Lightness: the value of Y
- Saturation: distance from the achromatic point
- Dominant wavelength: interception at the spectral loci of the line coming from the achromatic point and passing through the (x,y) point of the sample

A further shortcoming of the color space defined by the CIE is its non-uniformity: color distances do not correlate directly with the perception of the human eye and they cannot be used for the computation of color differences.

Color difference systems

The problem of non-uniformity of the CIE color diagram can be solved by applying a mathematical transformation that distorts the space to the extent that all colors are situated at the same distance. A further advantage of such systems is that there is a good direct correlation with the language used by colorists during color assessment.

CIE color space (CIELAB)

The CIE (L*a*b*) color space was introduced 1976 and is defined by the following equations:

$$L^* = 116 \cdot f\left(\frac{Y}{Y_n}\right) - 16 \text{ if } \frac{Y}{Y_n} > 0.008856$$

$$L^* = 903.3 \cdot \frac{Y}{Y_n} \qquad \text{if } \frac{Y}{Y_n} \le 0.008856$$

$$a^* = 500 \cdot \left[f\left(\frac{X}{X_n}\right) - f\left(\frac{Y}{Y_n}\right) \right]$$

$$b^* = 200 \cdot \left[f\left(\frac{Y}{Y_n}\right) - f\left(\frac{Z}{Z_n}\right) \right]$$

where the function *f(s)* is defined as:

$$f(s) = s^{1/3} \qquad \text{if } s > 0.008856$$

$$f(s) = 7.787 \cdot s + \frac{16}{116} \qquad \text{if } s \le 0.008856$$

$$\text{where } s = \left(\frac{X}{X_n}\right), \left(\frac{Y}{Y_n}\right) \text{ or } \left(\frac{Z}{Z_n}\right)$$

and (X_n, Y_n, Z_n) is the coordinate of the achromatic point, normally for the illuminant used.
While the CIELAB color space shows a high level of uniformity it is only capable to reproduce 80% of the samples used for validation; for this reason other color spaces have been proposed but none of them have been able to give a better performance.

CIELAB color difference formula

Based on the fact that the color space is now uniform a color difference formula can be given as the Euclidean distance between the coordinates of sample and standard:

$$\Delta E^*_{ab} = \sqrt{(\Delta L^*)^2 + (\Delta a^*)^2 + (\Delta b^*)^2}$$

where the quantities represent differences between corresponding coordinates of the two stimuli. Values of color differences can be used now to set color tolerances as a method to assess quality in color reproduction as:

ΔL*
+ = lighter
- = darker

Δa*
+ = redder
- = greener

Δb*
+ = yellower
- = bluer

Setting tolerance limits to each of the difference values above, results in small rectangular boxes arranged around the standard that fill out the color space. A problem arises when comparing this box-shaped tolerance model with the actual tolerances perceived by the human eye. In fact the latter resembles rather an ellipsoid than a box and this is the reason for the poor performance of the CIELAB tolerance box.

CIELCH color difference formula

Correlation of the language of the colorist and (L*a*b*) values leads to the definition of chroma (C*) and hue (h*) as follows:

$$C^* = \sqrt{a^{*2} + b^{*2}}$$

$$h^* = \arctan\left(\frac{b^*}{a^*}\right)$$

The color difference in this space is defined as:

$$\Delta E^*_{ab} = \sqrt{(\Delta L^*_{ab})^2 + (\Delta C^*_{ab})^2 + (\Delta H^*_{ab})^2}$$

where

$$\Delta H^*_{ab} = \sqrt{(\Delta a^*)^2 + (\Delta b^*)^2 - (\Delta C^*_{ab})^2}$$

and

$$\Delta C^*_{ab} = C^*_b - C^*_a$$

this represents another form to express tolerances as:

ΔL*	+ = lighter - = darker
ΔC*	+ = brighter - = duller
Δh*	hue difference

This corresponds to a conversion of a Cartesian coordinate system into a cylindrical one; setting tolerance values for the differences shown above will now result in circular section pieces that fill the color space. These circular section pieces approximate much closer an ellipsoid, resulting in a better correlation between observed and calculated color differences.

CMC color tolerance (pass/fail) equation

Continuing with the idea that an ellipsoid approximates much better the color tolerance system of the human eye a further equation was developed by the Colour Measuring Committee of the Society of Dyers and Coulourists in Great Britain and introduced in 1988. The CMC equation defines an ellipsoid with semi-axis corresponding to lightness, chroma and hue; the size and shape of the ellipsoid varies according the position of the coordinates in color space. The CMC equation is defined as:

$$\Delta E_{CMC} = \sqrt{\left(\frac{\Delta L^*_{ab}}{l \cdot S_l}\right)^2 + \left(\frac{\Delta C^*_{ab}}{c \cdot S_c}\right)^2 + \left(\frac{\Delta H^*_{ab}}{S_h}\right)^2}$$

where

$S_l = \frac{0.040975 \cdot L^*}{1 + 0.1765 \cdot L^*}$ if $L^* > 16$

otherwise $S_l = 0.511$

$$S_c = \frac{0.0638 \cdot \Delta C^*_{ab}}{1 + 0.131 \cdot \Delta C^*_{ab}} + 0.638$$

$$S_h = S_c \cdot (F \cdot T + 1 - F)$$

and

$$F = \sqrt{\frac{(\Delta C^*_{ab})^4}{(\Delta C^*_{ab})^4 + 1900}}$$

$T = 0.36 + |0.4 \cdot \cos(H_{ab} + 35)|$

unless $164° \leq H_{ab} \leq 345°$, then

$T = 0.56 + |0.2 \cdot \cos(H_{ab} + 168)|$

The equation contains two types of adjustable parameters:

- Commercial factor cf that multiplies the total value of the ellipse and varies its volume. This factor is used to make the total ellipsoid larger or smaller according to visual acceptance; if the value of cf=1 means that $\Delta E_{CMC}<1$ would pass but $\Delta E_{CMC}>1$ would fail.
- The factor (l:c): in general the human eye accepts a larger difference in lightness (l) than in chroma (c), therefore normally default numerical values of (2:1) are used for (l:c); these values may be altered to achieve better visual correlation with actual samples and set empirical color tolerances.

It must be remarked that the CMC equation builds a tolerance system and it is not a color space; in fact the practical application of the CMC equation aims to establish empirical color tolerances that match visual assessment rather than to seek absolute ones.

CIE94 color difference equation

Based on the good results of the CMC equation a further development was attempted by the CIE to improve the CIELCH color difference equation by introducing ellipsoids for building tolerances. The CIE94 equation is defined as:

$$\Delta E^*_{94} = \sqrt{\left(\frac{\Delta L^*_{ab}}{k_l \cdot S_l}\right)^2 + \left(\frac{\Delta C^*_{ab}}{k_c \cdot S_c}\right)^2 + \left(\frac{\Delta H^*_{ab}}{k_h \cdot S_h}\right)^2}$$

where

$$S_l = 1$$
$$S_c = 1 + 0.045 \cdot C^*_{ab\_s}$$
$$S_h = 1 + 0.015 \cdot C^*_{ab\_s}$$

with $C^*_{ab\_s}$ being the chroma coordinate of the standard. All k weighting factors are normally set to unity, but they may be altered by the user to achieve a better visual matching.

DIN99 color difference equation

The idea to correct the non-uniformity of CIELAB color space in order to preserve a Euclidean color difference has lead to the development of the DI99 color difference equation. Following definitions:

Red-green axis:
$$e = a^* \cdot \cos(16°) + b^* \cdot \sin(16°)$$

Yellow-blue axis:
$$f = 0.7 \cdot (-a^* \cdot \sin(16°) + b^* \cdot \cos(16°))$$

$G = \sqrt{e^2 + f^2}$ (chroma)

$h_{ef} = \arctan\left(\frac{f}{e}\right)$ (hue)

lead to a new system defined as:

$$L_{99} = 105.51 \cdot \frac{\ln(1 + 0.0158 \cdot L^*)}{k_e}$$
$$a_{99} = C_{99} \cdot \cos(h_{99})$$
$$b_{99} = C_{99} \cdot \sin(h_{99})$$

where:

$$C_{99} = \frac{\ln(1 + 0.045 \cdot G)}{0.045 \cdot k_{CH} \cdot k_e}$$
$$h_{99} = h_{ef} \cdot \frac{180}{\pi}$$

and k_{CH} and k_e are adjustable parameters.

The color difference formula is given then as the Euclidean distance:

$$\Delta E = \sqrt{(\Delta L_{99})^2 + (\Delta a_{99})^2 + (\Delta b_{99})^2}$$

and similarly:

$$\Delta E = \sqrt{(\Delta L_{99})^2 + (\Delta C_{99})^2 + (\Delta H_{99})^2}$$

thus recovering the uniformity of the color space as was the goal of the CIELAB color space. The adjustable parameters work in a similar way as with CMC equation and adopt generally the values (k_{CH}:k_e) = (2:0.5)

CIE DE 2000 color difference equation

Continuing the revision of older expressions for the color difference equation CIELAB94 has resulted in a totally new equation that has proven a high correlation with validation samples.
As with the DIN99 formula the procedure includes a partial correction of the color space in the axis a* given by:

$$a' = a^* \cdot (1 + G)$$

with

$$G = 0.5 \cdot \{1 - f(C^*_{ab,m})\}$$
$$f(C^*_{ab,m}) = \sqrt{\frac{C^{*7}_{ab,m}}{C^{*7}_{ab,m} + 25^7}}$$

and

$$C^*_{ab,m} = \frac{C^*_{ab,b} + C^*_{ab,s}}{2}$$

where *b*= sample and *s*= standard; values of L* and b* are not affected.
In this new space the definitions of color parameters are similar to CIELAB color space:

chroma: $C'_{ab} = \sqrt{a'^2 + b^{*2}}$

hue angle: $h'_{ab} = \arctan\left(\frac{b^*}{a'}\right)$

(add 360° if angle is negative) and

$$\Delta L'_{ab} = L*_{ab,b} - L*_{ab,s}$$

$$\Delta C'_{ab} = C'_{ab,b} - C'_{ab,s}$$

$$\Delta H'_{ab} = 2 \cdot \sin\left(\frac{\Delta h'_{ab}}{2}\right) \cdot \sqrt{C'_{ab,b} \cdot C'_{ab,s}}$$

$$\Delta h'_{ab} = h'_{ab,b} - h'_{ab,s}$$

(add 360° to the smaller hue angle if absolute value of difference is greater than 180°)

The CIE DE 2000 color difference formula is given then by:

$$\Delta E^*_{2000} = \sqrt{\left(\frac{\Delta L'_{ab}}{k_l \cdot S_l}\right)^2 + \left(\frac{\Delta C'_{ab}}{k_c \cdot S_c}\right)^2 + \left(\frac{\Delta H'_{ab}}{k_h \cdot S_h}\right)^2 + R_T \cdot \left(\frac{\Delta C'_{ab}}{k_c \cdot S_c}\right) \cdot \left(\frac{\Delta H'_{ab}}{k_h \cdot S_h}\right)}$$

with

$$S_L = 1 + 0.015 \cdot \frac{(L*_m - 50)^2}{\sqrt{20 + (L*_m - 50)^2}}$$

$$L*_m = \frac{L*_b + L*_s}{2}$$

$$S_C = 1 + 0.045 \cdot C'_m$$

where

$C'_m = \dfrac{C'_{ab,b} - C'_{ab,s}}{2}$ and

$h'_m = \dfrac{h'_{ab,b} + h'_{ab,s}}{2}$ (add 360° if absolute value of hue difference is greater than 180°)

$$S_H = 1 + 0.015 \cdot T \cdot C'_m$$

and

$$T = 1 - 0.17 \cdot \cos(h'_m - 30)$$
$$- 0.24 \cdot \cos(2 \cdot h'_m) + 0.32 \cdot \cos(3 \cdot h'_m + 6)$$
$$- 0.20 \cdot \cos(4 \cdot h'_m - 63)$$
$$R_T = -R_C \cdot \sin(2 \cdot \Delta\Theta)$$

with

$$\Delta\Theta = 30 \cdot \exp\left[-\left(\frac{h'_m - 275}{25}\right)^2\right]$$

and $R_C = 2 \cdot f(C'_m)$. All angles are in degree units

The additional term in the equation is a rotation of the ellipsoids in the blue region; as with the CIELAB94, all k weighting factors are normally set to unity,

In a general form an overall sensitivity factor can be introduced as k_e in a similar way as the commercial factor of the CMC equation:

$$\Delta V = \frac{\Delta E^*_{2000}}{k_e}$$

its value establish the overall size of the ellipsoid to adjust the difference formula to commercial needs.

Conclusions

Conceptually the development of color difference equations is quite advanced, however there is still no perfect agreement between observed and calculated color differences.

From the industrial point of view much of the needs are fulfilled matching the parameters of the CMC equation to local visual assessment; problems arise however during the communication of color differences and setting up specifications between suppliers and customers. One must always bear in mind that the main goal of the CMC equation is to set a pass/fail criterion.

Much work lies ahead before an agreement on a scientific basis is achieved. The new equations DIN99 and CIE DE 2000 appear quite promising but still they must prove their advantages in practical applications.

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